TOPOLOGICAL QUANTIZATION OF PHYSICAL PARAMETERS, GLOBAL ANOMALIES AND THE STOCHASTIC SCHEME

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The stochastic quantization scheme is shown *not* to enforce at finite stochastic time quantization of physical parameters in theories with multivalued actions. Unless these parameters are a priori quantized, stochastic averages do not possess an equilibrium limit. In particular, the stochastic scheme does not reproduce *global* anomalies at finite stochastic time.

Topological quantization of physical parameters is an important nonperturbative phenomenon occurring in some interesting models defined by multivalued actions such as:

(i) The chiral field model with the Wess–Zumino term [1,2] in even (euclidean) space–time dimensions D:

$$S_\xi^{(1)} = S_{\text{Chiral}}[U] + \mathrm{i}\xi\Gamma_{\text{WZ}}[U] ,$$

$$S_{\text{Chiral}}[U] = -\tfrac{1}{2}f^2 \int \mathrm{d}^D x\, \mathrm{tr}[L_\mu^2(U)] ,$$

$$L_\mu(U) = U^{-1}\partial_\mu U ,$$

$$\Gamma_{\text{WZ}}[U] = c_{D+1} \int_0^\infty \mathrm{d}x^{D+1} \int \mathrm{d}^D x\, \epsilon_{\mu_1 \ldots \mu_{D+1}} \times \mathrm{tr}[L_{\mu_1}(\widetilde{U}) \ldots L_{\mu_{D+1}}(\widetilde{U})] , \quad (1)$$

where

$$c_{D+1} \equiv -(\mathrm{i}/2\pi)^{(D+2)/2}(D/2)!\,[(D+1)!]^{-1} ,$$

and $\widetilde{U} = \widetilde{U}(x, x^{D+1})$ is a continuation of $U(x)$ from $\mathbf{R}^D$ to $\mathbf{R}^{D+1}$, $\widetilde{U}(x,0) = U(x)$, $\widetilde{U}(x, x^{D+1}) \to \mathbf{1}$ at infinity of $\mathbf{R}^{D+1}$.

(ii) Non-abelian gauge theories with Chern–Simons terms [3] in odd D:

$$S_\xi^{(2)} = S_{\text{YM}}[A] + \mathrm{i}\xi W_{\text{ChS}}[A] ,$$

$$S_{\text{YM}} = (4ng^2)^{-1} \int \mathrm{d}^D x\, \mathrm{tr}[F_{\mu\nu}^2(A)] , \quad (2)$$

$$W_{\text{ChS}}[A] = \int \mathrm{d}^D x\, \epsilon_{\mu_1 \ldots \mu_D} \times \mathrm{tr}[b_D A_{\mu_1} F_{\mu_2 \mu_3}(A) \ldots F_{\mu_{D-1}\mu_D}(A) - \ldots + (-\mathrm{i})^D c_D A_{\mu_1} \ldots A_{\mu_D}] , \quad (2\text{ cont'd})$$

where

$$b_D \equiv 2[((D+1)/2)!(4\pi)^{(D+1)/2}]^{-1} .$$

For a general construction and numerous examples of models with multivalued actions, see ref. [4].

Let us recall the reason for quantization of the parameter ξ:

$$\xi = 2\pi m , \quad m \in \mathbf{Z} . \quad (3)$$

The configuration spaces of the models (1), (2) possess a nontrivial topology [5–7]:

$$\mathcal{M}^{(1)} = \{U(x) \,|\, U\colon \mathbf{R}^D \to \mathrm{SU}(n),\ U(x) \to_{|x|\to\infty} \mathbf{1}\} ,$$

$$\pi_1(\mathcal{M}^{(1)}) = \pi_{D+1}(\mathrm{SU}(n)) = \mathbf{Z} \quad (\text{for even } D \leqslant 2n-2), \quad (4a)$$

$$\mathcal{M}^{(2)} = \mathcal{A}/\mathcal{G} , \quad \mathcal{G} = \mathcal{M}^{(1)} ,$$

$$\mathcal{A} = \{A_\mu(x) \,|\, A_\mu \to_{|x|\to\infty} -\mathrm{i}g^{-1}\partial_\mu g,\ \ g(x) \in \mathrm{SU}(n)\} . \quad (4b)$$

As a result, there exist *noncontractible* closed contours

$$\mathrm{C}^{(1)} = \{U(x;s) \,|\, U(x;0) = U(x;1) = U(x)\}$$

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in $\mathcal{M}^{(1)}$ and

$$C^{(2)} = \{A_\mu(x;s) | A_\mu(x;1) = u^{-1}(x)[A_\mu(x;0) - \mathrm{i}\partial_\mu]u(x), u \in \mathcal{G}\}$$

in $\mathcal{M}^{(2)}$ on which the actions (1), (2) are multivalued functionals:

$$\Gamma_{WZ}[U(\cdot;1)] = \Gamma_{WZ}[U(\cdot;0)] + N[\tilde{U}],$$

$$W_{ChS}[A_\mu(\cdot;1)] = W_{ChS}[A_\mu(\cdot;0)] + N[u];$$

$$N[u] = c_D \int \mathrm{d}^D x\, \epsilon_{\mu_1 \dots \mu_D} \operatorname{tr}[L_{\mu_1}(u) \dots L_{\mu_D}(u)], \quad (5)$$

where $N[\cdot]$ in (5) denote the corresponding topological charges. Thus, condition (3) is enforced in order to get well-defined quantum theories of (1), (2) within the functional integral formulation [1,3], i.e. to get single-valued Boltzmann weights $\exp(-S_\xi^{(1,2)})$.

Here we shall discuss in some detail the stochastic approach [8] to the quantization of theories with multivalued actions looking in this context for a mechanism for topological quantization of the relevant parameters.

In shorthand notations, the basic ingredients of this approach, the Langevin equations, read:

$$\partial_t \varphi = -\delta S_\xi / \delta\varphi + \eta,$$

$$\langle \eta(t,x)\eta(t',x')\rangle = 2\delta(t-t')\delta^{(D)}(x-x'), \quad (6)$$

where

$$S_\xi[\varphi(\cdot;1)] - S_\xi[\varphi(\cdot;0)] = \mathrm{i}\xi N, \quad N \in \mathbf{Z}, \quad (7)$$

for every noncontractible closed contour

$$C = \{\varphi(x;s) | \varphi(x;0) = \varphi(x;1) = \varphi(x)\} \quad (8)$$

in the configuration space $\mathcal{M}$. In particular, for the models (1), (2) one has

$$U^{-1}\partial_t U = -f^2 \partial_\mu L_\mu(U) - \mathrm{i}\xi(D+1)c_{D+1}\epsilon_{\mu_1\dots\mu_D} L_{\mu_1}(U) \dots L_{\mu_D}(U) + \eta,$$

$$\langle \eta^a(t,x)\eta^b(t',x')\rangle = 2\delta^{ab}\delta(t-t')\delta^{(D)}(x-x'),$$

$$\eta = \mathrm{i}T^a\eta^a \quad (9)$$

(the T^a are the hermitian generators of SU(n), $\operatorname{tr}(T^aT^b) = \delta^{ab}n$),

$$\partial_t A^a_\mu = -(1/g^2)(\nabla_\nu F_{\mu\nu}(A))^a - \tfrac{1}{2}(D+1)b_D \mathrm{i}\xi\epsilon_{\mu\nu_1\dots\nu_{D-1}} \times \operatorname{tr}(T^a F_{\nu_1\nu_2}\dots F_{\nu_{D-2}\nu_{D-1}}) + \eta^a_\mu,$$

$$\langle \eta^a_\mu(t,x)\eta^b_\nu(t',x')\rangle = 2\delta^{ab}\delta_{\mu\nu}\delta(t-t')\delta^{(D)}(x-x'). \quad (10)$$

In (10) the gauge-fixing "drift"-term [9] is suppressed since only gauge-invariant functionals of A_μ (i.e., functionals over $\mathcal{M}^{(2)}$ (4b)) will be considered. Our arguments below will apply to the formal unregularized stochastic scheme. Existing at present invariant regularizations (such as that of ref. [10]) will make the analysis of the equilibrium limit of the corresponding Fokker–Planck distribution difficult (eqs. (12), (13) below). Nevertheless, since the phenomena under consideration are topological in nature, it is assumed as in refs. [1,3] that they are not obscured by regularization of short-distance singularities.

The very simple, but crucial, observation is that unlike $S_\xi[\varphi]$ (cf. (7)) $\delta S_\xi/\delta\varphi$, standing on the RHS of (6), (9), (10), are smooth *single-valued* functionals for any ξ and, therefore, the Langevin equations (6), (9), (10) yield well-defined (after appropriate regularization) stochastic averages for *any* ξ:

$$\langle \mathcal{F}[\varphi^{(\xi)}(t,\cdot)]\rangle_\eta = \int \mathcal{D}\eta \exp\left(-\int \mathrm{d}t\, \mathrm{d}^D x\, \eta^2\right) \times \mathcal{F}[\varphi^{(\xi)}_\eta(t,\cdot)] = \int \mathcal{D}\varphi\, \mathcal{F}[\varphi]\, \mathcal{P}_\xi[\varphi;t], \quad (11)$$

where $\mathcal{F}$ is an arbitrary (gauge-invariant) functional of the solutions $\varphi^{(\xi)}_\eta(t,x)$ to (6), (9), (10) and $\mathcal{P}_\xi[\varphi;t]$ is the Fokker–Planck distribution:

$$\mathcal{P}_\xi[\varphi;t] = \int \mathcal{D}\eta \exp\left(-\int \mathrm{d}^D x\, \mathrm{d}t\, \eta^2\right) \times \prod_x \delta(\varphi(x) - \varphi^{(\xi)}_\eta(t,x)). \quad (12)$$

Thus one sees that no topological quantization of the parameter ξ is enforced at finite stochastic time t by the stochastic scheme for theories with multivalued actions. Then the important question arises as to how this topological quantization may emerge in terms of stochastic averages (11). As we are now going to show, although (11) make sense for any ξ, the equilibrium limit ($t \to \infty$) of (11) exists if and only if ξ is quantized apriori according to (3).

Indeed, the equilibrium distribution $\mathcal{P}_\xi[\varphi] = \lim_{t\to\infty} \mathcal{P}_\xi[\varphi; t]$ must satisfy the equation [8]

$$[\delta/\delta\varphi(x) + \delta S_\xi/\delta\varphi(x)]\, \mathcal{P}_\xi[\varphi] = 0\,, \tag{13}$$

whose obvious solution reads:

$$\mathcal{P}_\xi[\varphi] = \text{const.}\exp\Big(-\int_{C_{(\varphi,\varphi_0)}} \delta S_\xi/\delta\varphi\Big). \tag{14}$$

Here $C_{(\varphi,\varphi_0)}$ denotes an open path in the configuration space $\mathcal{M}$:

$$C_{(\varphi,\varphi_0)} = \{\varphi(x;s) | \varphi(x;0) = \varphi_0(x) - \text{reference point},$$
$$\varphi(x;1) = \varphi(x)\}\,,$$

and the functional line integral in (14) is defined as

$$\int_{C_{(\varphi,\varphi_0)}} (...) = \int_0^1 ds \int d^D x\, \partial_s \varphi(x;s)(...)\,.$$

Standard arguments imply that $\mathcal{P}_\xi[\varphi]$ (14) is a smooth path-independent, i.e. single-valued, equilibrium solution

$$\mathcal{P}_\xi[\varphi] = \text{const.}\exp\{-S_\xi[\varphi]\}\,,$$

and that, accordingly, the equilibrium limit

$$\lim_{t\to\infty} \langle \mathcal{F}[\varphi^{(\xi)}(t,\cdot)]\rangle_\eta = Z_\xi^{-1}\int \mathcal{D}\varphi\, \mathcal{F}[\varphi]\exp\{-S_\xi[\varphi]\} \tag{15}$$

exists, if and only if for every closed contour C (8) in $\mathcal{M}$:

$$\int_C \delta S_\xi/\delta\varphi = S_\xi[\varphi(\cdot;1)] - S_\xi[\varphi(\cdot;0)] = 2\pi i N\,,$$
$$N \in \mathbf{Z}\,. \tag{16}$$

Comparing (16) with (7) one finds that the topological quantization of ξ (3) is enforced only after invoking the requirement of the existence of the equilibrium limit (15).

Eq. (13) has a very transparent geometrical interpretation which directly follows from the general mathematical theory of models with multivalued actions [4]. It may be rewritten in the form

$$\mathcal{D}_x[\mathcal{A}]\,\mathcal{P}_\xi[\varphi] = 0\,,$$
$$\mathcal{D}_x[\mathcal{A}] \equiv \delta/\delta\varphi(x) + i\mathcal{A}_x[\varphi]\,,$$
$$\mathcal{A}_x[\varphi] \equiv -i\delta S_\xi/\delta\varphi(x)\,, \tag{13'}$$

where $\mathcal{D}_x[\mathcal{A}]$ is viewed as a functional covariant derivative with an abelian functional complex gauge potential $\mathcal{A}_x[\varphi]$:

$$\mathcal{A}_x[U] = \delta[\Gamma_{WZ} - iS_{Chiral}]/\delta\epsilon(x)\,,$$
$$\delta\epsilon(x) \equiv U^{-1}(x)\delta U(x)\,,$$
$$\mathcal{A}^a_{x,\mu}[A] = \delta[W_{ChS} - iS_{YM}]/\delta A^a_\mu(x)\,.$$

Then (16) is exactly the condition for global integrability of (13′) on the topological nontrivial configuration space $\mathcal{M}$, i.e. the condition that $\mathcal{A}_x[\varphi]$ is ***globally*** a "pure gauge". A similar interpretation of $\mathcal{A}^a_{x,\mu}[A] = \delta W_{ChS}[A]/\delta A^a_\mu(x)$ in a different context has appeared in refs. [11,7].

The same analysis applies for the SU(2) global chiral anomaly in $D = 4$ [12]:

$$\det[-i\nabla\!\!\!\!/_L(A^g)] = (-1)^{N_f}\det[-i\nabla\!\!\!\!/_L(A)]\,,$$

for N_f = odd number of fermion flavors, where $g(x)$ is a homotopically nontrivial SU(2) gauge transformation and $\nabla\!\!\!\!/_L(A)$ is the chiral Dirac operator. The corresponding effective action $S_{eff}[A] = S_{YM}[A] - N_f \ln\det[-i\nabla\!\!\!\!/_L(A)]$ is multivalued on $\mathcal{M}^{(2)}$ (4b) (cf. (7), (16)):

$$S_{eff}[A^g] = S_{eff}[A] - i\pi N_f\,.$$

Therefore, there is no way to detect the SU(2) global anomaly in the stochastic averages at finite t ($\delta S_{eff}/\delta A^a_\mu$ entering the pertinent Langevin equation (6) is gauge-covariant, i.e. single-valued). Accordingly, the latter does not possess an equilibrium limit.

Finally, let us add the following remarks. A further serious drawback of the stochastic scheme was found in ref. [13]. Namely, stochastic quantization of massless fermions in *odd D* interacting with (background) gauge fields does not reproduce the pertinent parity-violating anomalies [14], since it yields at finite t as well as in the equilibrium limit a gauge- and parity-covariant induced fermion current [13]. Therefore, the stochastic scheme is applicable to massless fermions in odd D only in those cases when the corresponding parity-anomalies can be cancelled by appropriate counterterms [15].

Unlike the above, there are *no* problems for the stochastic scheme to correctly reproduce the standard (perturbative) anomalies of chiral fermions in even D both at finite t [‡1] as well as in the equilibrium limit [17,18,13].

‡1 A statement [16] contradicting the latter result is criticized in ref. [17].

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